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ADMISSIBLE FORMS OF PLASTICITY RELATIONS FROM THE VIEWPOINT OF THE UNIQUENESS THEOREM

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At present, there are a great number of mathematical elastoplastic-medium models which are different in form and content [1, 2]. In constructing constitutive relations, each additional-loading region is divided into several nonintersecting regions in each of which stress–strain relations are of a differential-linear character. Cases are possible where strain (or stress) increments undergo a discontinuity in going from one to another additional-loading region and when there is no local potential in the additional-loading regions.

At the same time, there are restrictions on admissible forms of plasticity relations, which follows, for example, from Klushnikov's macrodeterminism postulate: (1) the local potential in each additional-loading region; (2) the continuity of a transition from one to another additional-loading region. These restrictions are such that they are "not satisfied by the endochronic plasticity theory, by many versions of the constitutive relations in the theory of elastoplastic processes, and by the sliding-type theories" [3, 4]. In essence, among the theories known for the time being, only the Reiss–Laning plastic-flow theory satisfies these restrictions.

Despite the restrictions mentioned above, they are disregarded with reference to the fact that plasticity should take into account the order of application of the load, i.e., the existence of a local potential is not obligatory. This forces us to consider again the derivations of the proposed restrictions to exclude any doubts in the correctness of these derivations. In this connection, we would like to consider the uniqueness theorem for solution of the elastoplastic problem in the small, namely, to consider the constitutive relations in a general enough form and the boundary-value elastoplastic problem, and to find, from the uniqueness of the solution of this problem, restrictions on admissible forms of the constitutive relations of plasticity that were indicated above.

To implement our plan, we have to choose a mathematical plasticity model. We assume that this model is such that there are only two additional-loading regions separated by a certain plane with the equation

$$f_n = f(\Delta\sigma_{kl}(t), \sigma_{pq}(t_n)) = 0 \tag{1}$$

(the introduction of only two additional-loading regions does not lead to loss of generality). We also assume that the function f_n is linear with respect to $\Delta\sigma_{kl}(t)$ and is not decreasing relative to $\sigma_{pq}(t_n)$. Here t is the load parameter which varies from 0 to T, $t_0 = 0 < t_1 < t_2 \ldots < t_N = T$ are the division points for the interval $[0, T]$, and $\Delta\sigma_{kl}(t) = \sigma_{kl}(t) - \sigma_{kl}(t_n)$ $(t_n \leqslant t \leqslant t_{n+1})$. The determining plasticity relations in the additional-loading regions are of the form

$$\Delta\varepsilon_{ij}(t) = A_{ijkl}(\sigma_{pq}(t_n))\Delta\sigma_{kl}(t) \quad \text{for} \quad f_n = f(\Delta\sigma_{kl}(t), \sigma_{pq}(t_n)) > 0, \tag{2}$$

$$\Delta\varepsilon_{ij}(t) = B_{ijkl}(\sigma_{pq}(t_n))\Delta\sigma_{kl}(t) \quad \text{for} \quad f_n = f(\Delta\sigma_{kl}(t), \sigma_{pq}(t_n)) \leqslant 0, \tag{3}$$

where A_{ijkl} and B_{ijkl} are the pliability-tensor components which are constant for $t_n \leqslant t < t_{n+1}$ and varying together with f_n at moment $t = t_{n+1}$, and $\Delta\varepsilon_{ij}(t) = \varepsilon_{ij}(t) - \varepsilon_{ij}(t_n)$. In the general case, $A_{ijkl} \neq A_{klij}$ and $B_{ijkl} \neq B_{klij}$. We call conditionally the regions in which relations (2) and (3) are satisfied regions P and Y.

Now we begin to solve the elastoplastic problem. We consider the static problem for a reinforcing material. The problem is formulated as follows: let the principal stress-strain state of a body, which corresponds

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to the loads $\mathbf{p}(t_n)$ specified on one part of the boundary S and to the displacement vectors $\mathbf{u}(t_n)$ specified on the other part of this boundary, be known at moment $t = t_n$. Then the stresses and the displacements vectors vary at moments t $(t_n \leqslant t \leqslant t_{n+1})$ and reach the values $\mathbf{p}(t_{n+1})$ and $\mathbf{u}(t_{n+1})$ at moment $t = t_{n+1}$. It is necessary to determine the stress-strain state increment which corresponds to the load $\Delta\mathbf{p}(t_{n+1}) = \mathbf{p}(t_{n+1}) - \mathbf{p}(t_n)$ and displacement-vector $\Delta\mathbf{u}(t_{n+1}) = \mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)$ increments.

After this problem is solved, the principal stress-strain state is summed with the increment obtained, and we have a new principal state, i.e., the situation repeats. One should note that a transition from state $\mathbf{p}(t_n)$, $\mathbf{u}(t_n)$ to state $\mathbf{p}(t_{n+1})$, $\mathbf{u}(t_{n+1})$ occurs at each point of the body surface in a Δ-tube, i.e., this transition can, generally speaking, vary to a certain extent, which is connected with technical capabilities of loading systems. It is clear that these variations should not introduce any significant correlations into the increment of the body's stress-strain state at step $\Delta t = t_{n+1} - t_n$, otherwise the solution of any problems becomes problematic. This circumstance should be taken into account in solving the initial problem via the constitutive relations for a medium.

We consider the formal proof of the uniqueness theorem when relations (2) and (3) are used. Note that this proof coincides, in outline, with that given by Koiter [5], Hill [6], and Ivlev and Bykovtsev [7], where the uniqueness of the solution of the elastoplastic problem was studied in the case of application of the plastic-flow theory.

For example, we assume that two solutions of the elastoplastic problem are obtained: $\Delta\sigma^1_{ij}$, Δu^1_i and $\Delta\sigma^2_{ij}$, Δu^2_i which satisfy the same boundary conditions. Let us check whether it is possible that $\Delta\sigma^1_{ij} \neq \Delta\sigma^2_{ij}$ and $\Delta u^1_i \neq u^2_i$ within the framework of (2) and (3). We represent the strain region V as a sum of four regions: $V = V_1 + V_2 + V_3 + V_4$. The initial stress-strain state is the same in both cases. We assume that an active additional loading occurs in the region V_1 for both solutions, the plastic-strain increment occurs for the first solution and the increment of only the elastic strain occurs in region V_2 for the second solution, elastic unloading and active loading occur in region V_3 for the first and second solutions, respectively, and the strain increments are elastic in region V_4 for both solutions. We write the differences, i.e., the solutions, and denote them by $\Delta\sigma_{ij} = \Delta\sigma^1_{ij} - \Delta\sigma^2_{ij}$ and $\Delta u_i = \Delta u^1_i - \Delta u^2_i$. For the solution, we have

$$0 = \int_S \Delta\sigma_{ij} n_j \Delta u_i \, dS = \int_V \Delta\sigma_{ij}\Delta\varepsilon_{ij}\, dV \quad (\Delta\varepsilon_{ij} = \Delta\varepsilon^1_{ij} - \Delta\varepsilon^2_{ij}). \tag{4}$$

Representing the volume integral as a sum of four integrals over regions V_1, V_2, V_3, and V_4, we obtain

$$\int_V \Delta\sigma_{ij}\Delta\varepsilon_{ij}\, dV = \int_{V_1} (\Delta\sigma^{1P}_{ij} - \Delta\sigma^{2P}_{ij})(\Delta\varepsilon^{1P}_{ij} - \Delta\varepsilon^{2P}_{ij})\, dV + \int_{V_2} (\Delta\sigma^{1P}_{ij} - \Delta\sigma^{2Y}_{ij})(\Delta\varepsilon^{1P}_{ij} - \Delta\varepsilon^{2Y}_{ij})\, dV$$

$$+ \int_{V_3} (\Delta\sigma^{1Y}_{ij} - \Delta\sigma^{2P}_{ij})(\Delta\varepsilon^{1Y}_{ij} - \Delta\varepsilon^{2P}_{ij})\, dV + \int_{V_4} (\Delta\sigma^{1Y}_{ij} - \Delta\sigma^{2Y}_{ij})(\Delta\varepsilon^{1Y}_{ij} - \Delta\varepsilon^{2Y}_{ij})\, dV. \tag{5}$$

The superscripts P and Y in the expressions $\Delta\sigma^1_{ij}$, $\Delta\sigma^2_{ij}$, and $\Delta\varepsilon^1_{ij}$, $\Delta\varepsilon^2_{ij}$ refer to the region to which the additional load is directed. Since $\Delta\varepsilon^{1P}_{ij} = A_{ijkl}\Delta\sigma^{1P}_{kl}$ and $\Delta\varepsilon^{2P}_{ij} = A_{ijkl}\Delta\sigma^{2P}_{kl}$, we have $\Delta\varepsilon^{1P}_{ij} - \Delta\varepsilon^{2P}_{ij} = A_{ijkl}(\Delta\sigma^{1P}_{kl} - \Delta\sigma^{2P}_{kl})$. Similarly, we have $\Delta\varepsilon^{1Y}_{ij} - \Delta\varepsilon^{2Y}_{ij} = B_{ijkl}(\Delta\sigma^{1Y}_{kl} - \Delta\sigma^{2Y}_{kl})$.

We study relation (5). For uniqueness of solution, it suffices to show the positiveness of the integrands in each integral that was given above. To do this in the first and last integrals, it is necessary and sufficient that the tensors $A_{(ijkl)}$ and $B_{(ijkl)}$ be positive define. Here $A_{ijkl} = A_{(ijkl)} + A_{[ijkl]}$, $B_{ijkl} = B_{(ijkl)} + B_{[ijkl]}$, $A_{(ijkl)}$, $B_{(ijkl)}$ and $A_{[ijkl]}$, $B_{[ijkl]}$ are the symmetric and skew-symmetric parts of the tensors A_{ijkl} and B_{ijkl}.

Using the representation of $\Delta\varepsilon^P_{ij} = (A_{ijkl} - B_{ijkl})\Delta\sigma^P_{kl} + B_{ijkl}\Delta\sigma^P_{kl}$, we transform (5) into the form

$$\int_V \Delta\sigma_{ij}\Delta\varepsilon_{ij} dV = \int_V \Delta\tau_{ij} B_{ijkl}\Delta\tau_{kl} dV + \int_{V_1} \Delta\tau_{ij}(A_{ijkl} - B_{ijkl})\Delta\tau_{kl} dV$$

$$+ \int_{V_2} (\Delta\sigma^{1P}_{ij} - \Delta\sigma^{2Y}_{ij})(A_{ijkl} - B_{ijkl})\Delta\sigma^{1P}_{kl} dV + \int_{V_3} (\Delta\sigma^{2P}_{ij} - \Delta\sigma^{1Y}_{ij})(A_{ijkl} - B_{ijkl})\Delta\sigma^{2P}_{ij} dV, \tag{6}$$

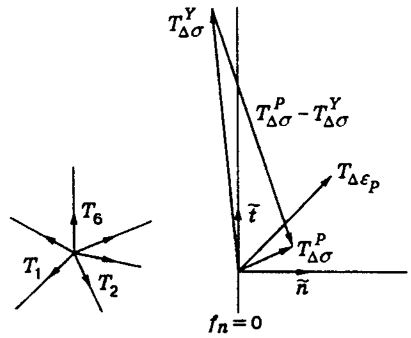

Fig. 1

where $\Delta\tau_{ij} = \Delta\sigma^{1P}_{ij} - \Delta\sigma^{2P}_{ij}$ in volume V_1, $\Delta\tau_{ij} = \Delta\sigma^{1P}_{ij} - \Delta\sigma^{2Y}_{ij}$ in volume V_2, $\Delta\tau_{ij} = \Delta\sigma^{1Y}_{ij} - \Delta\sigma^{2P}_{ij}$ in volume V_3, and $\Delta\tau_{ij} = \Delta\sigma^{1Y}_{ij} - \Delta\sigma^{2Y}_{ij}$ in volume V_4.

We now consider relation (6). For positiveness of the integrand in the second integral of (6) on the right-hand side, it is necessary that the tensor $A_{(ijkl)} - B_{(ijkl)}$ be positive define. To analyze the integrands in the third and fourth integrals of (6), we introduce the vector representation of tensors. In this connection, we have to give some explanations. Let $T_1, T_2, \ldots, T_6$ be the orthonormalized tensor bases [8, 9]. For two tensors, for example, for stress and strain tensors, the scalar product is found according to the formula $(T_\sigma, T_\varepsilon) = \sigma_{ij}\varepsilon_{ij}$ introduced by Novozhilov [8]. Therefore, it becomes clear how to define the orthogonality and orthonormalization notions. For two tensors, for example, for T_σ and T_ε, one can introduce the notions of their lengths and the angle between them. In this case, the formula

$$\cos\alpha = \frac{(T_\sigma, T_\varepsilon)}{|T_\sigma|\,|T_\varepsilon|} \tag{7}$$

is valid, where α is the angle between T_σ and T_ε. Using the representation (7), we analyze the sign of the integrands in the third and fourth integrals. In the given Euclidean tensor space, we give the plane $f_n = 0$, the tensors $T^Y_{\Delta\sigma}$ and $T^P_{\Delta\sigma}$, and the plastic-strain increment tensor (see Fig. 1): $T_{\Delta\varepsilon_P} = (A_{ijkl} - B_{ijkl}) \cdot\cdot\, T^P_{\Delta\sigma}$.

We introduce the unit tensors $\tilde{n}$ and $\tilde{t}$ directed along the normal and tangentially to the plane $f_n = 0$. Let the tensor $T_{\Delta\varepsilon_P}$ have a nonzero projection in the direction $\tilde{t}$. It is of interest whether the situation is possible where the angle between the tensors $T_{\Delta\varepsilon_P}$ and $T^P_{\Delta\sigma} - T^Y_{\Delta\sigma}$ is obtuse or, what is the same, the integrands are negative in the integrals indicated above (see Fig. 1). Clearly, the scalar product of the tensors $T_{\Delta\varepsilon_P}$ and $T^P_{\Delta\sigma} - T^Y_{\Delta\sigma}$ will be positive if and only if the tensor $T_{\Delta\varepsilon_P}$ is directed along the normal to the surface $f_n = 0$, i.e., the equality $(T_{\Delta\varepsilon_P}, \tilde{t}) = 0$ should occur and, hence, we obtain the plastic-flow theory. For increments of plastic strains, we separate the orthogonal eigenbasis coinciding with the basis $\tilde{n}$, $\tilde{t}$ and, therefore, we have

$$A_{[ijkl]} - B_{[ijkl]} = 0. \tag{8}$$

Henceforth, the tensor B_{ijkl} is symmetric ($B_{[ijkl]} = 0$), which is connected with the energy considerations in elasticity. As a result, using (8), we obtain $A_{[ijkl]} = 0$.

Thus, considering the uniqueness of the solution of the elastoplastic problem in the small, we obtain the same restrictions on the allowed forms of plasticity relations that were proposed by Klushnikov [3]. Under these restrictions, all the integrands in (6) are positive. For the expression on the right-hand side to be zero, it is necessary and sufficient that all $\Delta\tau_{ij}$ be equal to zero. This means that the volumes V_2 and V_3 should be zero. Thus, the uniqueness theorem in the small for a reinforcing body is considered.

Remark 1. The application of the deformation theory of plasticity, which is written in increments, leads to nonunique solutions of the elastoplastic problem, because there is a discontinuity in the strain-increment tensor in the direction of tensor $\tilde{t}$. The deformation theory of plasticity can be used for solution of elastoplastic problems if it is written in full stresses and strains.

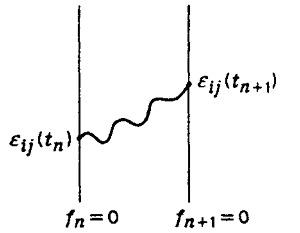

Fig. 2

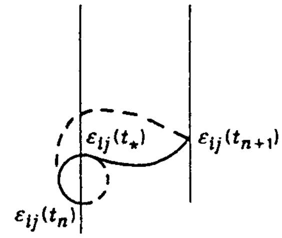

Fig. 3

Remark 2. Relations (2) and (3) are true if the additional-loading paths connecting two close points in the stress or strain space do not intersect the plane $f_n = 0$, i.e., they are of the form shown in Fig. 2.

If the loading path in the interval of load-parameter variation from $t = t_n$ to $t = t_{n+1}$ intersect the plane $f_n = 0$, as in Fig. 3 (intersections can be several in number), we obtain the following relations to define the strain increments $\Delta\varepsilon_{ij}(t_{n+1})$:

$$\Delta\varepsilon_{ij}(t_{n+1}) = \int\limits_{t_n}^{t_{n+1}} d\Delta\varepsilon_{ij}(t) = \Delta\varepsilon_{ij}(t)\Big|_{t_n}^{t_*} + \Delta\varepsilon_{ij}(t)\Big|_{t_*}^{t_{n+1}}, \quad t_n < t_* < t_{n+1}.$$

It is evident that in this case we have, instead of (2) and (3), the relations between the strain and stress increments which include the magnitudes of the discontinuities of the strain or stress increments in passing through the boundary separating the regions P and Y. The number of discontinuities is determined by variations in loads $\mathbf{p}(t)$ and displacements $\mathbf{u}(t)$ about the specified loading programs. This circumstance is a source of the nonuniqueness of the solution of boundary-value problems as well. To make the relations between stress and strain increments independent of the path of additional loading, which connects two infinitely close points in the stress–strain space, it is necessary and sufficient that they go continuously from relations in one additional-loading region to those in another region.

REFERENCES

1. I. V. Knets, *Modern Trends in the Mathematical Theory of Plasticity* [in Russian], Zinatne, Riga (1971).
2. A. A. Vakulenko, "The link between microproperties and macroproperties in elastoplastic media," *Itogi Nauki Tekh., Ser. Mekh. Deform. Tverd. Tela*, **22**, 3–54 (1991).
3. V. D. Klushnikov, "Theory of plasticity: state of the art and prospects," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 2, 103–116 (1993).
4. R. A. Vasin, "Constitutive relations of plasticity theory," *Itogi Nauki Tekh., Ser. Mekh. Deform. Tverd. Tela*, **21**, 3–75 (1990).
5. W. T. Koiter, "General theorems for elastic-plastic solids," *Prog. Solid Mech., Amsterdam*, **1**, 165–221 (1960).
6. R. Hill, *Mathematical Theory of Plasticity*, Oxford University Press, (1950).
7. D. D. Ivlev and G. I. Bykovtsev, *The Theory of a Reinforcing Plastic Body* [in Russian], Nauka, Moscow (1971).
8. V. V. Novozhilov, "On the forms of the relationship between stresses and strains in initially isotropic inelastic bodies (geometrical aspect of the problem)," *Prikl. Mat. Mekh.*, **27**, No. 5, 794–799 (1963).
9. Ya. Rykhlevskii, "On Hooke's law," *Prikl. Mat. Mekh.*, **48**, No. 3, 420–435 (1984).